RELATIONS

DEFINITION: A **relation** R on a set X is a set $R \subset X \times X$. If $(a, b) \in R$, we write $a\,R\,b$.

EXAMPLE:

- Take $X = \mathbb{Z}$ and Let R be $<$. Then $(1, 2) \in R$ since $1 < 2$.
- Let X be a set and define R on $\mathcal{P}(X)$ as follows: $A\,R\,B$ iff $A \subseteq B$.
  Then $\emptyset\,R\,A$ for all subsets A of X because $\emptyset \subseteq A$.

DEFINITION: A relation R on a set X is said to be:

- **reflexive** iff $x\,R\,x$ for all $x \in X$
- **symmertic** iff $x\,R\,y \implies y\,R\,x$
- **transitive** iff $x\,R\,y$ and $y\,R\,z \implies x\,R\,z$

EXAMPLE:

- $<$ on $\mathbb{Z}$ is transitive but neither reflexive nor symmetric.
- $\leq$ on $\mathbb{Z}$ is reflexive and transitive but not symmetric.
- $=$ on $\mathbb{Z}$ is reflexive, symmetric, and transitive.

EXAMPLE: Let $X = \{a, b, c\}$. Construct a set of ordered pairs $R \subseteq X \times X$ (if possible!) which is:

1. neither reflexive nor symmetric nor transitive.
2. reflexive but neither symmetric nor transitive.
3. symmetric but neither reflexive nor transitive.
4. transitive but neither reflexive nor symmetric.
5. reflexive and symmetric but not transitive.
6. reflexive and transitive but not symmetric.
7. symmetric and transitive but not reflexive.
8. reflexive, symmetric, and transitive.

EQUIVALENCE RELATIONS

DEFINITION: A relation R in X is an **equivalence relation** on X iff R is reflexive, symmetric, and transitive.

If $x \in X$ and R is an equivalence relation, $[x]_R = \{y \in X \, : \, x \, R \, y\}$ is the **equivalence class of** x **under** R.

EXAMPLE: Let $x, y \in \mathbb{Z}$ and define R as follows: $x \, R \, y$ iff $x - y$ is a multiple of 6.

- Prove R is an equivalence relation.
- Calculate $[0]_R$ and $[5]_R$.
- How many different equivalence classes are there?

EXAMPLE: Suppose X is a set and R is an equivalence relation.

Prove for all $x, y \in X$, either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$.

NOTE: This says equivalence relations **partition** a set into a pairwise disjoint cover: $X = \biguplus_{x \in X} [x]_R$

Likewise, given a partition of a set, you can define an equivalence relation whose equivalence classes are the members of the partition! (Do you see how?)

DEFINITION: Given a set X and an equivalence relation R, the set X **mod** R, denoted X/R is

$$X/R = \{[x]_R \, : \, x \in X\}$$

That is, X/R is the set of equivalence classes of X under R.

EXAMPLE: Let $x, y \in \mathbb{Z}$ and define R as follows: $x \, R \, y$ iff $x - y$ is a multiple of 6. Find $\mathbb{Z}/R$.

EXAMPLE: Suppose $F : X \to Y$ is a function. If $a, b \in X$, define $a \, R \, b$ iff $F(a) = F(b)$.

- Show R is an equivalence relation.
- In this case, we'll write X/R as X/F. What is X/F in this case?

DEFINITION: Let X be a set and let R be an equivalence relation on X.

The map $q : X \to X/R$ given by $q(x) = [x]_R$ is called the **quotient map** of X onto X/R.

EXAMPLE: Prove the quotient map is a surjective function.

QUOTIENT SPACES

DEFINITION: Let $(X, \mathcal{T})$ be a space and let R be an equivalence relation on X.

Let $q : X \to X/R$ be the quotient map.

The **quotient topology** $\mathcal{T}_R$ is defined as:

$$\mathcal{T}_R = \left\{ U \subseteq X/R \, : \, q^{-1}(U) \in \mathcal{T} \right\}$$

NOTE: If you look back at our first notes on continuous functions, you'll find we've already shown:

- $\mathcal{T}_R$ a topology.
- Equipping X/R with $\mathcal{T}_R$ makes q continuous.
- $\mathcal{T}_R$ is the coarsest (finest) topology we can put on X/R to make q continuous.

EXAMPLE: Let $X = [0, 2\pi]$ with the Euclidean subspace topology. Let $R = \{(x, x) \, : \, x \in X\} \cup \{(0, 2\pi)\}$

What does X/R 'look' like?

THEOREM: Suppose $F : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous.

Let $(X/F, \mathcal{T}_F)$ denote the quotient space created from the relation: $a \, R \, b$ iff $F(a) = F(b)$ with quotient map q.

Then there is s a unique continuous function $\overline{F} : (X/F, \mathcal{T}_F) \to (Y, \mathcal{U})$ such that $\overline{F} \circ q = F$.